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General Section

On newforms and Saito-Kurokawa lifts

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ABSTRACT

In this paper, we derive the Saito-Kurokawa isomorphism on the space of newforms for Maaß spezialschar of degree 2, weight k, level M, where $32|M$ and primitive character χ modulo M with $\chi(-1) = (-1)^k$ and χ^2 is primitive modulo $M/2$. We first develop the corresponding theory of newforms for respective spaces of half-integral weight modular forms, Jacobi forms and then for Maass forms and as a consequence we get the Saito-Kurokawa isomorphism.

1. Introduction

Through numerical examples N. Kurokawa [9], independently H. Saito and H. Maass predicted the existence of Hecke eigenforms in the space of degree two Siegel modular

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forms whose eigenvalues correspond to the eigenvalues of elliptic modular forms. H. Maass introduced and studied a canonical subspace called 'Maaß spezialschar' in degree two Siegel modular forms. The Maass space corresponds to the elliptic modular forms via Saito-Kurokawa correspondence. It lifts a given normalised newform f of weight $2k-2$, level M and character χ^2 into Hecke eigenforms F in the Maass space of degree two, weight k, level M and character χ. In general for a given normalised newform f, how many linearly independent Maass Hecke eigenforms F are lifted into the Maass space is not known. If M is a square free integer and χ is the trivial character then the Saito-Kurokawa correspondence gives an isomorphism and hence the normalised newform f is lifted into a unique Hecke eigenform F. For the details we refer to [1,15–17,21,4,14,12,6]. The present paper aims to set up isomorphism for the Saito-Kurokawa correspondence on newforms for arbitrary level M under the assumptions that $32|M$ and the involved characters are primitive, and prove that the newform f is lifted into 2 linearly independent Maass forms under the Saito-Kurokawa correspondence.

We organise the paper in the following manner. In §3, we consider both plus spaces of the same weight $k-1/2$ and character χ_0 for the groups $\Gamma_0(M), \Gamma_0(4M)$, respectively. We then derive the Shimura-Kohnen lifts on each of the spaces and develop the respective theory of newforms for both the spaces. Next, in §4 we derive the Eichler-Zagier canonical map and get the theory of newforms for the space of Jacobi forms. Finally, we develop the theory of newforms for the Maass space in §5.

More precisely, we make the following assumptions. Let $k \geq 2$, $M = 2^{\alpha-2}N$ $(2 \nmid N,\ \alpha > 6)$ be integers. Let χ modulo M be a Dirichlet character with $\epsilon = \chi(-1)$ such that $\chi_0 = \left(\frac{4\epsilon}{\cdot}\right)\chi$ is an even character modulo M. Let $cond(\chi) = M$ and $cond(\chi^2) = M/2$. If f is a normalised newform of weight $2k-2$, level $M/2$ with character χ^2, we derive that the form f is lifted into two linearly independent Hecke eigenforms $F, F|B_S(4)$ in the Maass space of degree two Siegel modular forms of weight k, level M with character χ under the Saito-Kurokawa correspondence (see Theorem 1.4 and Remark 5.2 in §5.1). Moreover, the relevant Saito-Kurokawa isomorphism maps the space of newforms $S_{2k-2}^{new}(M/2, \chi^2)$ into the space of newforms $S_k^{*,new}(\Gamma_0^2(M), \chi)$ in the Maass space.

Let $S_{k-1/2}(M, \chi_0)$ denote the space of cusp forms of weight $k-1/2$ for $\Gamma_0(M)$ and character χ_0. We denote the Petersson scalar product by

$$\langle f, g\rangle = \frac{1}{i_M} \int\limits_{\Gamma_0(M)\backslash\mathbb{H}} f(\tau)\overline{g(\tau)} y^{k-1/2}\frac{dxdy}{y^2},$$

where $\tau = x + iy$, $y > 0$, i_M denotes the index of $\Gamma_0(M)$ in $SL_2(\mathbb{Z})$ and $f, g \in S_{k-1/2}(M, \chi_0)$. Let $S_{k-1/2}^{+}(M, \chi_0)$ denote the Kohnen plus space which consists of the cusp forms in $S_{k-1/2}(M, \chi_0)$ whose n-th Fourier coefficients vanish unless $\epsilon(-1)^{k-1}n \equiv 0, 1 (\text{mod } 4)$.

We state the following dimension equality (see Lemma 3.7): If $k \geq 2$, $cond(\chi) = M$ and $cond(\chi^2) = M/2$,

$$\dim S_{k-1/2}(M,\chi_0)=\dim S_{2k-2}(M/2,\chi^2). \tag{1}$$

If $D\equiv 1(4)$ is a fundamental discriminant with $\epsilon(-1)^{k-1}D>0$, we consider the D-th Shimura-Kohnen lift S_D which is the same as the D-th Shimura lift, given by

$$g|S_D=\sum_{n\geq 1}\left(\sum_{d|n}\chi(d)\left(\frac{D}{d}\right)d^{k-2}a_g\left(\frac{|D|n^2}{d^2}\right)\right)e^{2\pi inz},$$

where $g\in S_{k-1/2}(M,\chi_0)$. In order to set up the Shimura lifts S_D on $S^+_{k-1/2}(M,\chi_0)$, we consider the image of m-th Poincaré series in the plus space $S^+_{k-1/2}(M,\chi_0)$ under S_D and derive its explicit image as a period function in $S_{2k-2}(M/2,\chi^2)$. By varying the integers $m\geq 1$ with $\epsilon(-1)^{k-1}m\equiv 0,1(\text{mod } 4)$, we get that all the Poincaré series $P^+_{k-1/2,M,\chi_0;m}$ span the space $S^+_{k-1/2}(M,\chi_0)$. Hence, the Shimura-Kohnen lifts S_D maps $S^+_{k-1/2}(M,\chi_0)$ into $S_{2k-2}(M/2,\chi^2)$ and the adjoint Shintani lifts S^*_D maps $S_{2k-2}(M/2,\chi^2)$ into $S^+_{k-1/2}(M,\chi_0)$. If $f\in S_{2k-2}(M/2,\chi^2)$ is a normalised Hecke eigenform, it is known that there exists a fundamental discriminant D ($\epsilon(-1)^{k-1}D>0$, $(D,M)=1$) such that the special value $L(f,\overline{\chi}\left(\frac{D}{\cdot}\right),k-1)\neq 0$ (see the remark after Theorem 1.1 in Chapter 6, [18]). Therefore, using the condition $cond(\chi)=M$ and following the computations as in ([8], p. 137), we derive that the $|D|$-th Fourier coefficient of $f|S^*_D$ is non-zero. Since this is valid for each of the normalised Hecke eigenform $f\in S_{2k-2}(M/2,\chi^2)$ and by using the dimension equality as given by the equation (1), we observe that the space $S^+_{k-1/2}(M,\chi_0)$ coincides with the full space $S_{k-1/2}(M,\chi_0)$. We state the theory of newforms for the space $S^+_{k-1/2}(M,\chi_0)$:

Theorem 1.1. *Let $k\geq 2$, $32|M$, $cond(\chi)=M$ and $cond(\chi^2)=M/2$. Then,*

$$S^+_{k-1/2}(M,\chi_0)=S_{k-1/2}(M,\chi_0).$$

There exists a finite linear combination of Shimura lifts ψ_K which defines an isomorphism from $S^+_{k-1/2}(M,\chi_0)$ into $S_{2k-2}(M/2,\chi^2)$. In particular, we have the strong multiplicity one theorem on $S^+_{k-1/2}(M,\chi_0)$.

Next, we consider the spaces $S^+_{k-1/2}(4M,\chi_0)$ and $S_{2k-2}(M,\chi^2)$. The condition $cond(\chi^2)=M/2$ along with the dimension formula and the theory of newforms for $S_{2k-2}(M,\chi^2)$ gives $S^{new}_{2k-2}(M,\chi^2)=\{0\}$. To get the same for the plus space $S^+_{k-1/2}(4M,\chi_0)$, we derive that the Shimura lifts S_D ($D\equiv 0,1(\text{mod } 4),\epsilon(-1)^{k-1}D>0,(D,M)=1$) map $S^+_{k-1/2}(4M,\chi_0)$ into $S_{2k-2}(M,\chi^2)$. Then we find the relation between the spaces $S^+_{k-1/2}(M,\chi_0)$ and $S^+_{k-1/2}(4M,\chi_0)$. We state the following main theorem for modular forms of half-integral weight (see §3.2):

Theorem 1.2.

$$S^{+,new}_{k-1/2}(4M,\chi_0)=\{0\} \ \ and \ \ S^{new}_{2k-2}(M,\chi^2)=\{0\},$$

$$S^{+}_{k-1/2}(4M,\chi_0)=S^{+}_{k-1/2}(M,\chi_0)\bigoplus S^{+}_{k-1/2}(M,\chi_0)|B_4,$$

$$S_{2k-2}(M,\chi^2)=S_{2k-2}(M/2,\chi^2)\bigoplus S_{2k-2}(M/2,\chi^2)\big|B_2.$$

Moreover, the isomorphism ψ_K maps $S^{+}_{k-1/2}(4M,\chi_0)$ into $S_{2k-2}(M,\chi^2)$.

In order to get the Maass lift and theory of newforms for the Maass space, we let $k\geq 2$, $\epsilon=(-1)^k$ and we first develop the theory of newforms for the space of Jacobi cusp forms $J^{cusp}_{k,1}(M,\chi)$. This can be done by deriving the Eichler-Zagier canonical map (see preliminaries for definition)

$$\mathcal{Z}_1 : J^{cusp}_{k,1}(M,\chi)\longrightarrow S^{+}_{k-1/2}(4M,\chi_0).$$

This is an isomorphism preserving the Hecke eigenforms and the scalar product structures (see Proposition 4.2 and Lemma 4.3). Using Theorem 1.2 as above and the existence of Eichler-Zagier canonical isomorphism we get the following. There exists a non-zero cusp form $\phi\in J^{cusp}_{k,1}(M,\chi)$ such that $\phi|B_J(4)$ belongs to $J^{cusp}_{k,1}(M,\chi)$ and $\phi|\mathcal{Z}_1\in S^{+}_{k-1/2}(M,\chi_0)$. We call the inverse image of the space $S^{+}_{k+1/2}(M,\chi_0)$ as the space of newforms in $J^{cusp}_{k,1}(M,\chi)$. Let $P_{k,1,M,\chi_0;D,r}$ denote the (D,r)-th Poincaré series in $J^{cusp}_{k,1}(M,\chi)$; let $U_J(4)$ and $B_J(4)$ be operators on $J^{cusp}_{k,1}(M,\chi)$ (see preliminaries for definitions). Let $J^{cusp;new}_{k,1}(M,\chi)$ denote the linear span over complex numbers of the set $\{P_{k,1,M,\chi;D,r}|U_J(4) : D\}$, where D varies over all the discriminants with $4\big|D$ and $D/4\equiv 0,1 (\mathrm{mod}\ 4)$. We call the elements of an orthogonal basis consisting of simultaneous eigenforms under all the Hecke operators $T_J(n)$ in $J^{cusp;new}_{k,1}(M,\chi)$ as Jacobi newforms of weight k, index 1, level M and character χ. We have (see, Lemma 4.7):

$$J^{cusp;new}_{k,1}(M,\chi)|\mathcal{Z}_1=S^{+}_{k-1/2}(M,\chi_0).$$

We now state the following main theorem for the theory of newforms of Jacobi cusp forms:

Theorem 1.3.

$$J^{cusp}_{k,1}(M,\chi)=J^{cusp;new}_{k,1}(M,\chi)\bigoplus J^{cusp;new}_{k,1}(M,\chi)|B_J(4).$$

The space $J^{cusp;new}_{k,1}(M,\chi)$ is isomorphic to the space $S_{2k-2}(M/2,\chi^2)$ under a certain linear combination of Shimura lifts. Hence, The multiplicity one result holds good on $J^{cusp;new}_{k,1}(M,\chi)$.

Let $\mathcal{S}_k(\Gamma_0^2(M),\chi)$ denote the space of degree two Siegel cusp forms of level M and character χ, where χ is primitive Dirichlet character modulo M as above. Let $\phi \in J_{k,1}^{cusp}(M,\chi)$. Let $\iota_{M,\chi}$ denote the Maass embedding (as in [6]) defined by

$$\phi|\iota_{M,\chi}(\tau,z,w) = \sum_{m=1}^{\infty}(\phi|_{k,1}V_{m,\chi})(\tau,z)e^{2\pi imw},$$

where $V_{m,\chi}$ is the index shifting operator on $J_{k,1}^{cusp}(M,\chi)$ (defined in §5). Then by Theorem 3.2 of [6], the map $\iota_{M,\chi}$ is an embedding from $J_{k,1}^{cusp}(M,\chi)$ to $\mathcal{S}_k(\Gamma_0^2(M),\chi)$. Denote the image of $J_{k,1}^{cusp}(M,\chi)$ under this embedding by $\mathcal{S}_k^*(\Gamma_0^2(M),\chi)$. We define the space of Maass newforms by

$$\mathcal{S}_k^{*;new}(\Gamma_0^2(M),\chi) := J_{k,1}^{cusp;new}(M,\chi)|\iota_{M,\chi}.$$

Finally, combining the above results we get the relevant Saito-Kurokawa lifting for level M and a primitive character χ modulo M. For a comprehensive theory on Saito-Kurokawa correspondence we refer to [4,6]. Let $f \in S_{2k-2}(M/2,\chi^2)$ be a normalised newform. Let $F \in \mathcal{S}_k^{*;new}(\Gamma_0^2(M),\chi)$ be the associated unique (up to a scalar) newform. The corresponding Andrianov zeta function $Z_F(s)$ (see §5.1) has an Euler product expansion

$$Z_F(s) = \prod_{p|M}(1-\mu_p p^{-s})^{-1}\prod_{p\nmid M} Q_p(p^{-s})^{-1},$$

where

$$\begin{aligned}Q_p(p^{-s}) = 1 - \gamma_p p^{-s} + (p\omega_p + (p^2+1)\chi(p^2)p^{2k-5})p^{-2s} - \omega_p\chi(p^2)p^{2k-3-3s}\\ + \chi(p^4)p^{4k-6-4s}.\end{aligned}$$

Now, using the isomorphisms $\psi_k, \mathcal{Z}_1, \iota_{M,\chi}$, and Theorems 1.1 to 1.3, we derive the following:

Theorem 1.4.

$$\mathcal{S}_k^*(\Gamma_0^2(M),\chi) = \mathcal{S}_k^{*;new}(\Gamma_0^2(M),\chi)\bigoplus \mathcal{S}_k^{*;new}(\Gamma_0^2(M),\chi)|B_S(4).$$

The multiplicity one theorem is valid on $\mathcal{S}_k^{*;new}(\Gamma_0^2(M),\chi)$. Also, $\mathcal{S}_k^{*;new}(\Gamma_0^2(M),\chi)$ is in one to one correspondence with $S_{2k-2}(M/2,\chi^2)$ under the Saito-Kurokawa isomorphism. A given normalised Hecke eigenform $f \in S_{2k-2}(M/2,\chi^2)$ is lifted into two equivalent Hecke eigenforms $F, F|B_S(4)$, where $F \in \mathcal{S}_k^{*;new}(\Gamma_0^2(M),\chi)$ is the newform satisfying

$$Z_F(s) = L(s-k+1,\chi)L(s-k+2,\chi)L(f,s).$$

2. Preliminaries

2.1. Modular forms of half-integral weight

Let (a, b) denote the *gcd* of given integers a, b. For complex numbers x and y with $y \neq 0$, define $e_y(x) := e^{2\pi ix/y}$. Let $\mathbb{H}$ denote the upper half plane. For a real number r, let $\lfloor r \rfloor$ denote the greatest integer less than or equal to r. Let $M_{k-1/2}(M, \chi_0)$ denote the space of modular forms of weight $k-1/2$ for $\Gamma_0(M)$ with character χ_0. Let $S_{k-1/2}(M, \chi_0)$ denote the space of cusp forms in $M_{k-1/2}(M, \chi_0)$. For $g \in M_{k-1/2}(M, \chi_0)$, we write its Fourier expansion at the cusp ∞ as

$$g(z) = \sum_{n \geq 0} a_g(n) e^{2\pi inz}.$$

If $m \geq 1$ is an integer, the operator B_m is defined on formal series by

$$B_m : \sum_{n \geq 0} a_g(n) e^{2\pi inz} \longrightarrow \sum_{n \geq 0} a_g(n) e^{2\pi inmz}.$$

Let G denote the collection of all ordered pairs $(A, \phi(z))$, where $A \in \mathrm{GL}_2^+(\mathbb{R})$ and $\phi(z)$ is a holomorphic function on $\mathbb{H}$ such that

$$\phi^2(z) = t\frac{cz + d}{\sqrt{\det(A)}}, \quad t \in \{\pm 1\}.$$

G forms a group under the group law:

$$(A, \phi(z))(B, \psi(z)) := (AB, \phi(Bz)\psi(z)).$$

For a complex valued function g defined on the upper half plane $\mathbb{H}$ and $(A, \phi(z)) \in G$, we define the stroke operator by

$$g\big|_{k-1/2}(A, \phi(z))(z) := \phi(z)^{-2k+1} g(Az).$$

Now let

$$\xi = \left(\begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}, \epsilon^{1/2} e^{\pi i/4} \right) \text{ and } \xi' = \left(\begin{pmatrix} 4 & -1 \\ 0 & 4 \end{pmatrix}, \epsilon^{-1/2} e^{-\pi i/4} \right).$$

Then, formal computations show that both $g|\xi$ and $g|\xi'$ belong to $M_{k-1/2}(M, \chi_0)$ for all $g \in M_{k-1/2}(M, \chi_0)$. Also, if g is a cusp form, $g|\xi$ and $g|\xi'$ are cusp forms. The projection operator is defined on $M_{k-1/2}(M, \chi_0)$ by

$$Pr_+ := \left(\frac{8}{2k-1} \right) \frac{1}{2\sqrt{2}\epsilon} (\xi + \xi') + \frac{1}{2} \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, 1 \right),$$

and we let $g|Pr_+$ for its image, where $g \in M_{k-1/2}(M, \chi_0)$. For this we refer to [11]. Define

$$M^+_{k-1/2}(M, \chi_0) := M_{k-1/2}(M, \chi_0)|Pr_+, \text{ and}$$
$$S^+_{k-1/2}(M, \chi_0) := S_{k-1/2}(M, \chi_0) \cap M^+_{k-1/2}(M, \chi_0).$$

Then, formal computations give

$$g|Pr_+ = \sum_{\substack{n \geq 0 \\ \epsilon(-1)^{k-1} n \equiv 0,1 (\bmod\ 4)}} a_g(n) e^{2\pi i n z},$$

where $g = \sum_{n \geq 0} a_g(n) e^{2\pi i n z}$. For an integer $n \geq 1, (n, M) = 1$, let T_{n^2} denote n-th Hecke operator on $M_{k-1/2}(4M, \chi_0)$. It preserves the space of cusp forms $S_{k-1/2}(4M, \chi_0)$. A non-zero form $g \in S^+_{k-1/2}(4M, \chi_0)$ is called a Hecke eigenform if it is a simultaneous eigenform for all the Hecke operators T_{n^2}, $(n, M) = 1$.

Let $S_{2k-2}(M/2, \chi^2)$ denote the space of cusp forms of weight $2k - 2$, level $M/2$ and character χ^2. Let T_n $((n, M) = 1)$ and $U_n(n|M)$ denote the Hecke operators, and W_{p^a} (p a prime, $p^a|M/2$ and $p^{a+1} \nmid M/2$) denote the W operator on $S_{2k-2}(M/2, \chi^2)$ as in [2,10]. By a normalised newform $f \in S_{2k-2}(M/2, \chi^2)$ we mean an eigenform for all the operators $T_n((n, M) = 1)$, $U_n(n|M)$ and W_{p^a} (p a prime, $p^a|M/2$ and $p^{a+1} \nmid M/2$) with $a_f(1) = 1$.

2.2. Jacobi forms

Let $\ell \geq 1$ be an integer. Let $J_{k,\ell}(M, \chi)$ denote the space of Jacobi forms of weight k, index ℓ and character χ, and its sub space of cusp forms is denoted by $J^{cusp}_{k,\ell}(M, \chi)$. We refer to [4] for the development of the theory of Jacobi forms. A Jacobi form $\phi \in J^{cusp}_{k,\ell}(M, \chi)$ has a Fourier expansion of the form

$$\phi(\tau, z) = \sum_{\substack{n, r \in \mathbb{Z}, \\ r^2 < 4n\ell}} c_\phi(n, r) e(n\tau + rz).$$

Let $n, r, n', r' \in \mathbb{Z}$ with $r^2 < 4n\ell, r'^2 < 4n'\ell$. Then we have $c_\phi(n, r) = c_\phi(n', r')$ if $r'^2 - 4n'\ell = r^2 - 4n\ell$ and $r' \equiv r (\bmod\ 2\ell)$. Thus, we write the Fourier expansion of ϕ as

$$\phi(\tau, z) = \sum_{\substack{D<0, r \in \mathbb{Z}, \\ D \equiv r^2 (\bmod\ 4n\ell)}} c_\phi(D, r) e\left(\frac{r^2 - D}{4\ell}\tau + rz\right).$$

We note that when the index is 1, $J^{cusp}_{k,1}(M, \chi) = \{0\}$ unless $\chi(-1) = (-1)^k$. So whenever we consider the space $J_{k.1}(M, \chi)$, we let $\chi(-1) = (-1)^k$.

The operators $U_J(4)$ and $B_J(4)$ on $J_{k,1}^{cusp}(M,\chi)$ are defined on formal series by

$$\sum_{\substack{D<0, r\in\mathbb{Z},\\ D\equiv r^2 (\mathrm{mod}\ 4)}} c_\phi(D,r) e\left(\frac{r^2-D}{4}\tau + rz\right)\Big| U_J(4)$$

$$= \sum_{\substack{D<0, r\in\mathbb{Z},\\ D\equiv r^2 (\mathrm{mod}\ 4)}} c_\phi(4D,2r) e\left(\frac{r^2-D}{4}\tau + rz\right) \text{ and}$$

$$\sum_{\substack{D<0, r\in\mathbb{Z},\\ D\equiv r^2 (\mathrm{mod}\ 4)}} c_\phi(D,r) e\left(\frac{r^2-D}{4}\tau + rz\right)\Big| B_J(4)$$

$$= \sum_{\substack{D<0, r\in\mathbb{Z},\\ D\equiv r^2 (\mathrm{mod}\ 16)}} c_\phi\left(\frac{D}{4},\frac{r}{2}\right) e\left(\frac{r^2-D}{4}\tau + rz\right).$$

If $\phi \in J_{k,1}^{cusp}(M,\chi)$, the Eichler-Zagier map $\mathcal{Z}_1$ on ϕ is given by

$$\sum_{\substack{D<0, r\in\mathbb{Z},\\ D\equiv r^2 (\mathrm{mod}\ 4)}} c_\phi(D,r) e\left(\frac{r^2-D}{4}\tau + rz\right) \longmapsto \sum_{\substack{D<0,\\ D\equiv r^2 (\mathrm{mod}\ 4)}} c_\phi(D,r) e(|D|\tau).$$

Let $\phi, \psi \in J_{k,1}^{cusp}(M,\chi)$. We define the Petersson scalar product by

$$\langle \phi, \psi\rangle = \frac{1}{i_M} \int_{\Gamma^J(M)\backslash\mathbb{H}\times\mathbb{C}} \phi(\tau,z)\overline{\psi(\tau,z)} e^{-4\pi y^2/v} v^{k-3} du dv dx dy,$$

where $\tau = u+iv, v>0$, $z = x+iy$ and $\Gamma^J(M) := \Gamma_0(M) \ltimes (\mathbb{Z}\times\mathbb{Z})$ is the Jacobi group of level M.

Let $D<0$ be a discriminant and $r (\mathrm{mod}\ 2\ell)$ with $r^2 \equiv D (\mathrm{mod}\ 4\ell)$. Then, we denote the (D,r)-th Poincaré series in $J_{k,\ell}^{cusp}(M,\chi)$ by $P_{k,\ell,M,\chi;D,r}$ and it is characterised by the relation

$$\langle \phi, P_{k,\ell,M,\chi;D,r}\rangle = \alpha_{k,\ell,D,M} C_\phi(D,r)$$

for all $\phi \in J_{k,\ell}^{cusp}(M,\chi)$, where

$$\alpha_{k,\ell,D,M} = \frac{\Gamma(k-3/2)}{\pi^{k-3/2} i_M} \ell^{k-2} |D|^{-k+3/2}.$$

3. Newforms for the spaces $S^+_{k-1/2}(M,\chi_0)$ and $S^+_{k-1/2}(4M,\chi_0)$

In this section, we develop the theory of newforms for both the spaces $S^+_{k-1/2}(M,\chi_0)$ and $S^+_{k-1/2}(4M,\chi_0)$, where $k \geq 2$ and $\chi(\text{mod } M)$ is primitive.

3.1. Theory of newforms of $S^+_{k-1/2}(M,\chi_0)$

Let $n \geq 1$ be an integer. Let $P_{k-1/2,M,\chi_0;n}$ denote the n-th Poincaré series in $S_{k-1/2}(M,\chi_0)$ ([13], page 238) characterised by

$$< g, P_{k-1/2,M,\chi_0;n} >= i_M^{-1}\frac{\Gamma(k-3/2)}{(4\pi n)^{k-3/2}}a_g(n) \quad \text{for all } g \in S_{k-1/2}(M,\chi_0).$$

Let

$$P^+_{k-1/2,M,\chi_0;n} = P_{k-1/2,M,\chi_0;n}\big|Pr_+.$$

Then using proposition 2 of [13], we have the Fourier expansion of $P^+_{k-1/2,M,\chi_0;n}$ in the following lemma.

Lemma 3.1. *Let n be a positive integer such that $\epsilon(-1)^{k-1}n \equiv 0,1(\text{mod } 4)$. We have*

$$P^+_{k-1/2,M,\chi_0;n}(\tau) = \sum_{\substack{m\geq 1 \\ \epsilon(-1)^{k-1}m\equiv 0,1(\text{mod } 4)}} g^+_{k-1/2,M,\chi_0;n}(m)e^{2\pi i m\tau}, \tag{2}$$

where

$$g^+_{k-1/2,M,\chi_0;n}(m) = \delta_{n,m} + \pi\sqrt{2}(-1)^{\lfloor\frac{k}{2}\rfloor}(1-(-1)^{k-1}i)(m/n)^{k/2-3/4} \times \sum_{c\geq 1} H_{Mc,\chi}(m,n)J_{k-3/2}\left(\frac{4\pi\sqrt{mn}}{Mc}\right),$$

$\delta_{n,m}$ is the Kronecker delta, $J_{k-3/2}(.)$ is the Bessel function and

$$H_{Mc,\chi}(m,n) = \frac{1}{Mc}\sum_{\delta(\text{mod } Mc)^*} \overline{\chi}_0(\delta)\left(\frac{Mc}{\delta}\right)\left(\frac{-4}{\delta}\right)^{k-1/2} e_{Mc}(m\delta+n\delta^{-1})$$

with $\delta^{-1} \in \mathbb{Z}$ and $\delta\delta^{-1} \equiv 1(\text{mod } Mc)$ is a Kloosterman type sum.

We state the following which we will use later.

Lemma 3.2. *With notations as above,*

$$g^+_{k-1/2,M,\chi_0;n}(m) = (m/n)^{k-3/2}g^+_{k-1/2,M,\overline{\chi}_0;m}(n). \tag{3}$$

3.1.1. A certain period function $F_{2k-2,M/2,\chi^2;|D|M^2m,D,\overline{\chi}}$

We first let $k > 2$ and derive the results. If $k = 2$, we mention appropriate changes later in order to get the results (see the paragraph before Lemma 3.7). Let $m \geq 1$ be an integer with $\epsilon(-1)^{k-1}m \equiv 0, 1(\text{mod } 4)$. Let D be an odd fundamental discriminant with $\epsilon(-1)^{k-1}D > 0$. Let $\mathcal{Q}_{M^2/4,|D|M^2m}$ be the set of all integral binary quadratic forms $Q(x, y) := ax^2+bxy+cy^2$ having discriminant $b^2-4ac = |D|M^2m$ and $a \equiv 0(\text{mod } M^2/4)$. If $Q = Q(x, y) \in \mathcal{Q}_{M^2/4,|D|M^2m}$, define the genus character χ_D (see, [7]) by $\chi_D(Q) = \left(\frac{D}{r}\right)$ or 0 according as $(a, b, c, D) = 1$ or not, where Q represents r. Define a period function in $S_{2k-2}(M/2, \chi^2)$ by

$$F_{2k-2,M/2,\chi^2;|D|M^2m,D,\overline{\chi}}(z) = \sum_{\substack{Q=[a,b,c],\\ Q\in\mathcal{Q}_{M^2/4,|D|M^2m}}} \overline{\chi(c)}\chi_D(Q)Q(z,1)^{-(k-1)}.$$

Similar function has been considered by Kohnen in [7] for trivial character and for odd M and for a generic case where the conductor of χ depends on the even part of the level, similar functions have been constructed in [13] in connection with Shimura correspondence. Note that

$$F_{2k-2,M/2,\chi^2;|D|M^2m,D,\overline{\chi}} \in S_{2k-2}(M/2, \chi^2).$$

For $f \in S_{2k-2}(M/2, \chi^2)$, let

$$r_{2k-2,M/2,\chi^2}(f; |D|M^2m, D, \chi) = \sum_{\substack{Q(\text{mod } \Gamma_0(M/2)),\\ |Q|=|D|M^2m,\\ 4a\equiv 0(\text{mod } M^2)}} \chi(c)\chi_D(Q) \int_{C_Q} f(z)d_{Q,k}z$$

where C_Q is the image in $\Gamma_0(M/2)\backslash\mathbb{H}$ of the semicircle $az^2 + bRe(z) + c = 0$ oriented from $(-b - \sqrt{|D|M^2m})/2a$ to $(-b + \sqrt{|D|M^2m})/2a$, if $a \neq 0$ or of the vertical line $bRe(z) + c = 0$ oriented from $-c/b$ to $i\infty$ if $b > 0$ and from $i\infty$ to $-c/b$, if $a = 0$, and $d_{Q,k}z = (az^2 - bz + c)^{k-2}dz$. Then, we have

Proposition 3.3 *(see: [7], proposition 7 and [13], proposition 4). For* $f \in S_{2k-2}(M/2, \chi^2)$,

$$\langle f(z), \overline{F_{2k-2,M/2,\chi^2;|D|M^2m,D,\overline{\chi}}(-\overline{z})}\rangle = \frac{\pi\, 2^{-2k+4}\binom{2k-4}{k-2}}{i_{M/2}(|D|M^2m)^{k-3/2}}\, r_{2k-2,M/2,\chi^2}(f; |D|M^2m, D, \chi).$$

We state the Fourier expansion of $F_{2k-2,M/2,\chi^2;|D|M^2m,D,\overline{\chi}}$ in the following:

Proposition 3.4 *(see: [13], proposition 1).*

$$F_{2k-2,M/2,\chi^2;|D|M^2m,D,\overline{\chi}}(z) = \sum_{n\geq 1} C_{2k-2,M/2,\chi^2;|D|M^2m,D,\overline{\chi}}(n; |D|M^2m)e^{2\pi inz}, \qquad (4)$$

where

$$
\begin{aligned}
&C_{2k-2,M/2,\chi^2;|D|M^2m,D,\overline{\chi}}(n;|D|M^2m)\\
&\quad= \frac{2(-2\pi)^{k-1}}{M^{k-3/2}(k-2)!}(n^2/m|D|)^{(k-2)/2}\Bigg\{(-1)^{\lfloor k/2\rfloor}\chi(n/\sqrt{m/|D|})\\
&\qquad\quad \times R_{\overline{\chi},D}\left(\frac{D}{n/\sqrt{m/|D|}}\right)\delta(n/\sqrt{m/|D|})|D|^{-1/2}\\
&\qquad + \pi\sqrt{2}(n^2/m|D|)^{1/4}\\
&\qquad\quad \times \sum_{\substack{a\geq 1,\\ M^2|4a}} a^{-1/2}S_{a,\overline{\chi}}(|D|M^2m,n)J_{k-3/2}\left(\frac{\pi n\sqrt{|D|M^2m}}{a}\right)\Bigg\},
\end{aligned}
$$

$R_{\overline{\chi},D}$ *is the Gauss sum given by*

$$
R_{\overline{\chi},D} = (M|D|)^{-1/2}\left(\frac{4\epsilon D}{-1}\right)^{-1/2}\sum_{r (\mathrm{mod}\ M|D|)}\overline{\chi}(r)\left(\frac{D}{r}\right)e_{M|D|}(r),
$$

$S_{a,\overline{\chi}}(|D|M^2m,n)$ *is the finite exponential sum given by*

$$
\begin{aligned}
&S_{a,\overline{\chi}}(|D|M^2m,n)\\
&\quad= \sum_{\substack{b (\mathrm{mod}\ 2a),\\ b^2\equiv|D|M^2m (\mathrm{mod}\ 4a)}} \overline{\chi}\left(\frac{b^2-|D|M^2m}{4a}\right)\chi_D\left(a,b,\frac{b^2-|D|M^2m}{4a}\right)e_{2a}(nb),
\end{aligned}
$$

and

$$
\delta(x) = \begin{cases} 1 & \text{if } x\in\mathbb{Z},\\ 0 & \text{otherwise.}\end{cases}
$$

3.1.2. Action of Shimura map on Poincaré series

To get

$$
S_{D;k-1/2,M,\chi_0} : S^+_{k-1/2}(M,\chi_0) \longrightarrow S_{2k-2}(M/2,\chi^2),
$$

we derive the image of Poincaré series under Shimura lifts.

Proposition 3.5.

$$
P^+_{k-1/2,M,\chi_0;m}|S_{D;k-1/2,M,\chi_0} = \lambda_{k,D,M,\chi}F_{2k-2,M/2,\chi^2;|D|M^2m,D,\overline{\chi}}, \tag{5}
$$

where

$$\lambda_{k,D,M,\chi} = \left(\frac{2(-2\pi)^{k-1}}{(k-2)!} (M|D|)^{-k+3/2} (-1)^{\lfloor k/2 \rfloor} R_{\overline{\chi},D} \right)^{-1}.$$

Proof. Let

$$P^{+}_{k-1/2,M,\chi_0;m} | S_{D;k-1/2,M,\chi_0} = \sum_{n \geq 1} b(n) e^{2\pi i n z}.$$

Using the equations (2) and (3) we get

$$\begin{aligned}
b(n) &= \sum_{d|n} \chi(d) \left(\frac{D}{d}\right) d^{k-2} (|D|n^2/d^2 m)^{k-3/2} \\
&\quad \times \left(\delta_{\frac{|D|n^2}{d^2},m} + \pi\sqrt{2}(-1)^{\lfloor \frac{k}{2} \rfloor}(1-(-1)^{k-1}i)(m/\frac{|D|n^2}{d^2})^{k/2-3/4} \right. \\
&\quad \left. \times \sum_{c\geq 1} H_{Mc,\overline{\chi}}(m, \frac{|D|n^2}{d^2}) J_{k-3/2}\left(\frac{4\pi\sqrt{m\frac{|D|n^2}{d^2}}}{Mc} \right) \right) \\
&= \left(\sum_{d|n} \chi(d) \left(\frac{D}{d}\right) d^{-k+1} (|D|n^2/m)^{k-3/2} \delta_{\frac{|D|n^2}{d^2},m} \right) \\
&\quad + \left(\pi\sqrt{2}(-1)^{\lfloor \frac{k}{2} \rfloor}(1-(-1)^{k-1}i) \right. \\
&\quad \times \sum_{d|n} \chi(d) \left(\frac{D}{d}\right) d^{k-2} (|D|n^2/d^2 m)^{k-3/2} (m/\frac{|D|n^2}{d^2})^{k/2-3/4} \\
&\quad \left. \times \sum_{c\geq 1} H_{Mc,\overline{\chi}}(m, \frac{|D|n^2}{d^2}) J_{k-3/2}\left(\frac{4\pi n\sqrt{m|D|}}{Mcd} \right) \right) \\
&= \left(\chi(n/\sqrt{m/|D|}) \left(\frac{D}{n/\sqrt{m/|D|}}\right) (n/\sqrt{m/|D|})^{-k+1} (|D|n^2/m)^{k-3/2} \delta(n/\sqrt{m/|D|}) \right) \\
&\quad + \left(\pi\sqrt{2}(-1)^{\lfloor \frac{k}{2} \rfloor}(1-(-1)^{k-1}i) \right. \\
&\quad \times \sum_{d|n} \chi(d) \left(\frac{D}{d}\right) d^{k-2} (|D|n^2/d^2 m)^{k-3/2} (m/\frac{|D|n^2}{d^2})^{k/2-3/4} \\
&\quad \left. \times \sum_{c\geq 1} H_{Mc,\overline{\chi}}(m, \frac{|D|n^2}{d^2}) J_{k-3/2}\left(\frac{4\pi n\sqrt{m|D|}}{Mcd} \right) \right).
\end{aligned} \tag{6}$$

Substituting (6) and (4) into left- and right-hand sides of equation (5) respectively, it is enough to prove that,

$$\left(\chi(n/\sqrt{m/|D|})\left(\tfrac{D}{n/\sqrt{m/|D|}}\right)(n/\sqrt{m/|D|})^{-k+1}(|D|n^2/m)^{k-3/2}\delta(n/\sqrt{m/|D|})\right)$$
$$+\Bigg(\pi\sqrt{2}(-1)^{\lfloor\frac{k}{2}\rfloor}(1-(-1)^{k-1}i)$$
$$\times\textstyle\sum_{d|n}\chi(d)\left(\frac{D}{d}\right)d^{k-2}(|D|n^2/d^2m)^{k-3/2}(m/\frac{|D|n^2}{d^2})^{k/2-3/4}$$
$$\times\textstyle\sum_{c\geq 1}H_{Mc,\overline{\chi}}(m,\frac{|D|n^2}{d^2})J_{k-3/2}\left(\frac{4\pi n\sqrt{m|D|}}{Mcd}\right)\Bigg)$$
$$=\Bigg(\lambda_{k,D,M,\chi}\tfrac{2(-2\pi)^{k-1}}{(k-2)!}M^{-k+3/2}(n^2/m|D|)^{(k-2)/2}$$
$$\times(-1)^{\lfloor k/2\rfloor}R_{\overline{\chi},D}\,\chi(n/\sqrt{m/|D|})\left(\tfrac{D}{n/\sqrt{m/|D|}}\right)\delta(n/\sqrt{m/|D|})|D|^{-1/2}\Bigg)$$
$$+\Bigg(\lambda_{k,D,M,\chi}\tfrac{2(-2\pi)^{k-1}}{(k-2)!}M^{-k+3/2}(n^2/m|D|)^{(k-2)/2}$$
$$\times\pi\sqrt{2}(n^2/m|D|)^{1/4}\sum_{\substack{a\geq 1,\\ M^2|4a}}a^{-1/2}S_{a,\overline{\chi}}(|D|M^2m,n)J_{k-3/2}\left(\tfrac{\pi n\sqrt{|D|M^2m}}{a}\right)\Bigg). \tag{7}$$

The first terms on both sides of equation (7) vanishes if $m/|D|$ is not a square of an integer and if $n \neq \sqrt{m/|D|}$. Suppose this happens and if $n = \sqrt{m/|D|}$, then both first terms are equal in the above equation. Now, we compare the second terms on both sides of (7). Substituting $cd = a$ on the left-hand side of (7), it is enough to prove that

$$\pi\sqrt{2}(-1)^{\lfloor\frac{k}{2}\rfloor}(1-(-1)^{k-1}i)\sum_{d|(a,n)}\chi(d)\left(\frac{D}{d}\right)d^{k-2}(|D|n^2/d^2m)^{k-3/2}(m/\frac{|D|n^2}{d^2})^{k/2-3/4}$$
$$\times\sum_{a\geq 1}H_{Ma/d,\overline{\chi}}(m,\frac{|D|n^2}{d^2})J_{k-3/2}\left(\frac{4\pi n\sqrt{m|D|}}{Ma}\right)$$
$$=\lambda_{k,D,M,\chi}\frac{2(-2\pi)^{k-1}}{(k-2)!}M^{-k+3/2}(n^2/m|D|)^{(k-2)/2}\pi\sqrt{2}(n^2/m|D|)^{1/4}\sum_{a\geq 1}((M/2)^2a)^{-1/2}$$
$$\times S_{a(M/2)^2,\overline{\chi}}(M^2|D|m,n)J_{k-3/2}\left(\frac{4\pi n\sqrt{m|D|}}{Ma}\right),$$

i.e.,

$$\pi\sqrt{2}(-1)^{\lfloor\frac{k}{2}\rfloor}(1-(-1)^{k-1}i)\sum_{d|(a,n)}\chi(d)\left(\frac{D}{d}\right)d^{-1/2}|D|^{k/2-3/4}n^{k-3/2}m^{-k/2+3/4}$$
$$\times\sum_{a\geq 1}H_{Ma/d,\overline{\chi}}(m,\frac{|D|n^2}{d^2})J_{k-3/2}\left(\frac{4\pi n\sqrt{m|D|}}{Ma}\right)$$

$$= ((-1)^{\lfloor \frac{k}{2} \rfloor} R_{\overline{\chi},D})^{-1} |D|^{k/2-3/4} n^{k-3/2} m^{-k/2+3/4} \pi\sqrt{2} \sum_{a\geq 1} ((M/2)^2 a)^{-1/2}$$

$$\times S_{a(M/2)^2,\overline{\chi}}(M^2|D|m,n) J_{k-3/2}\left(\frac{4\pi n\sqrt{m|D|}}{Ma}\right),$$

which follows from the proposition stated below and whose proof needs a standard set of arguments. Hence we omit the details; for a proof, we refer to ([7], proposition 5; [13], proposition 3).

Proposition 3.6. *For all $a, m, n \in \mathbb{N}$, we have*

$$S_{aM^2/4,\overline{\chi}}(M^2|D|m,n)$$
$$= R_{\overline{\chi},D}\sqrt{aM^2/4}(1-(-1)^{k-1}i) \sum_{d|(a,n)} \chi(d)\left(\frac{D}{d}\right) d^{-1/2} H_{Ma/d,\overline{\chi}}(m, \frac{|D|n^2}{d^2}). \quad \square$$

Let D, $\epsilon(-1)^{k-1}D > 0$ be a fundamental discriminant. The adjoint of Shimura map $S_{D;k-1/2,M,\chi_0}$ is given by

$$S^*_{D;2k-2,M/2,\chi^2} : S_{2k-2}(M/2,\chi^2) \longrightarrow S^+_{k-1/2}(M,\chi_0).$$

We now let $k = 2$. All the stated results in the above subsection are still valid. We define the required period function when $k = 2$, by using the standard 'Hecke trick' as in [7]. We leave the details, since we need to proceed along the same lines of arguments of the quoted paper by Kohnen. We make the following observation. Let $k = 2$ and N be an arbitrary odd integer. The proof of Theorem 2 of [7] shows that the Shintani map S^*_D maps the first Poincaré series $P_{2,N;1} \in S_2(N)$ into $|D|$-th Poincaré series $P^+_{3/2,4N;|D|} \in S^+_{3/2}(4N)$. These results are also valid in our case.

We now prove Theorem 1.1. We compare the dimensions of the required spaces under the stated assumptions. We have

Lemma 3.7.

$$\dim S_{k-1/2}(M,\chi_0) = \dim S_{2k-2}(M/2,\chi^2) = \frac{1}{2}\dim S_{2k-2}(M,\chi^2).$$

Proof. Let $k > 2$. Using the notations as in [19], we have

$$\dim S_{k-1/2}(M,\chi_0) = \frac{(k-3/2)2^{\alpha-2}N}{12} . \frac{3}{2}\prod_{p|N}(1+1/p) - \frac{\zeta(k-1/2, 2^{\alpha-2}N,\chi_0)}{2}\prod_{p|N} 2,$$
$$= (2k-3)2^{\alpha-6}N\prod_{p|N}(1+1/p) - 2^{\nu(N)}$$

$$\dim S_{2k-2}(M/2,\chi^2) = \frac{(2k-3)2^{\alpha-3}N}{12}\cdot\frac{3}{2}\prod_{p|N}(1+1/p) - \frac{\lambda(r_2,s_2,2)}{2}\prod_{p|N}2$$
$$= (2k-3)2^{\alpha-6}N\prod_{p|N}(1+1/p) - 2^{\nu(N)},$$

where $\nu(N)=$ number of distinct primes dividing N. Hence, $\dim S_{k-1/2}(M,\chi_0) = \dim S_{2k-2}(M/2,\chi^2)$. In a similar manner we can verify

$$\dim S_{2k-2}(M/2,\chi^2) = \frac{1}{2}\dim S_{2k-2}(M,\chi^2).$$

This completes the proof of the lemma for the case $k>2$. The proof for $k=2$ follows using the same computations combined with the following facts. Since $cond(\chi)=M$ the space $M_{1/2}(M,\chi_0)$ becomes trivial, which follows from the work of Serre-Stark (see the Theorem A in [20]). Similarly, since χ is not the trivial character, we also observe that both the spaces $M_0(M/2,\chi^2)$ and $M_0(M,\chi^2)$ are trivial (see, for example, the Theorem 7.4.1 of [3]). □

Proof of Theorem 1.1. Let $d=\dim S_{2k-2}(M/2,\chi^2)=\dim S_{k-1/2}(M,\chi_0)$ and $\{f_1,f_2,\ldots,f_d\}$ be the orthogonal basis of normalised Hecke eigenforms of $S_{2k-2}(M/2,\chi^2)$. For each $i, 1\le i\le d$ select a normalised Hecke eigenform $f(=f_i)$ and then a fundamental discriminant $D(=D_i)$ with $\epsilon(-1)^{k-1}D>0$, $(D,M)=1$ and $L(f,\overline{\chi}\left(\frac{D}{\cdot}\right),k-1)\neq 0$. Since $|D|$-th Fourier coefficient of $f|S^*_{D;2k-2,M,\chi}$ is equal to a non-zero constant multiple of $L(f,\overline{\chi}\left(\frac{D}{\cdot}\right),k-1)$, by using the arguments which give the Theorem 4.1 of [8], we derive that the d cusp forms $f_i|S^*_{D_i;2k-2,M,\chi}$ constructed as above forms an orthogonal set in $S^+_{k-1/2}(M,\chi_0)$.

Thus we select an orthogonal set of cusp forms $\{g_i : 1\le i\le d\}$ in $S^+_{k-1/2}(M,\chi_0)$ such that

$$g_i|S_{D;k-1/2,M,\chi_0} = a_{g_i}(|D|)f_i \quad \text{and} \quad \frac{f_i}{<f_i,f_i>}|S^*_{D;2k-2,M/2,\chi^2} = \overline{a_{g_i}(|D|)}\frac{g_i}{<g_i,g_i>}$$

hold good.

Now $S^+_{k-1/2}(M,\chi_0)$ is a subspace of $S_{k-1/2}(M,\chi_0)$ having $d=\dim S_{k-1/2}(M\chi_0)$ linearly independent cusp forms, and hence we conclude that

$$S^+_{k-1/2}(M,\chi_0) = S_{k-1/2}(M,\chi_0).$$

This completes the proof of Theorem 1.1. □

3.2. Theory of newforms of $S^+_{k-1/2}(4M,\chi_0)$

Let $n\ge 1$ be an integer and let $P_{k-1/2,4M,\chi_0;n}$ denote the n-th Poincaré series in $S_{k-1/2}(4M,\chi_0)$ characterised by

$$< g, P_{k-1/2,4M,\chi_0;n} >= i_{4M}^{-1} \frac{\Gamma(k-3/2)}{(4\pi n)^{k-3/2}} a_g(n) \quad \text{for all } g \in S_{k-1/2}(4M, \chi_0).$$

Let

$$P^+_{k-1/2,4M,\chi_0;n} = P_{k-1/2,4M,\chi_0;n} | Pr_+.$$

Let $k > 2$ and $F_{2k-2,M,\chi^2;|D|M^2m,D,\overline{\chi}}$ denote the function defined by

$$F_{2k-2,M,\chi^2;|D|M^2m,D,\overline{\chi}}(z) = \sum_{\substack{Q=[a,b,c], \\ Q \in \mathcal{Q}_{M^2,|D|M^2m}}} \overline{\chi(c)} \chi_D(Q) Q(z,1)^{-(k-1)}.$$

Note that $F_{2k-2,M,\chi^2;|D|M^2m,D,\overline{\chi}} \in S_{2k-2}(M, \chi^2)$.

If $k = 2$, then we define Poincaré series and the period functions in $S_2(M, \chi^2)$ as defined by Kohnen in [7] and the stated results in this subsection are valid.

We have the following result, when $k \geq 2$ and with other conditions assumed on M, χ and χ^2. The proof follows using similar arguments as in the proof of Proposition 3.5:

Proposition 3.8.

$$P^+_{k-1/2,4M,\chi_0;m} | S_{D;k-1/2,4M,\chi_0} = \lambda_{k,D,M,\chi} F_{2k-2,M,\chi^2;|D|M^2m,D,\overline{\chi}},$$

where

$$\lambda_{k,D,M,\chi} = \left(\frac{2(-2\pi)^{k-1}}{(k-2)!} (M|D|)^{-k+3/2} (-1)^{\lfloor k/2 \rfloor} R_{\overline{\chi},D} \right)^{-1}.$$

Hence, image of $P^+_{k-1/2,4M,\chi_0;m}$ under the map $S_{D;k-1/2,4M,\chi_0}$ is a constant multiple of the period function $F_{2k-2,M,\chi^2;|D|M^2m,D,\overline{\chi}}$. Since $F_{2k-2,M,\chi^2;|D|M^2m,D,\overline{\chi}} \in S_{2k-2}(M, \chi^2)$, and all the Poincaré series $P^+_{k-1/2,4M,\chi_0;m}$ span the space $S^+_{k-1/2}(4M, \chi_0)$ we conclude that

$$S_{D;k-1/2,4M,\chi_0} : S^+_{k-1/2}(4M, \chi_0) \longrightarrow S_{2k-2}(M, \chi^2).$$

We now observe the following.

Lemma 3.9. $S^{new}_{2k-2}(M, \chi^2) = \{0\}$.

Proof. Since $cond(\chi^2) = M/2$, the theory of newforms developed by W. Li in [10] gives

$$S_{2k-2}(M, \chi^2) = (S_{2k-2}(M/2, \chi^2) \oplus S_{2k-2}(M/2, \chi^2)|B_2) \bigoplus S^{new}_{2k-2}(M, \chi^2).$$

Now B_2 is an injective linear map and since $\dim S_{2k-2}(M, \chi^2) = 2 \dim S_{2k-2}(M/2, \chi^2)$, we get

$$S^{new}_{2k-2}(M, \chi^2) = \{0\}. \quad \square$$

We define

$$S^{+,old}_{k-1/2}(4M,\chi_0) = S^{+}_{k-1/2}(M,\chi_0) \oplus S^{+}_{k-1/2}(M,\chi_0)|B_4$$

and $S^{+,new}_{k-1/2}(4M,\chi_0)$ as the orthogonal complement of $S^{+,old}_{k-1/2}(4M,\chi_0)$ with respect to the Petersson scalar product. We have

Lemma 3.10. $S^{+,new}_{k-1/2}(4M,\chi_0) = \{0\}$.

Proof. We first observe the direct sum decomposition

$$S^{+}_{k-1/2}(4M,\chi_0) = S^{+}_{k-1/2}(M,\chi_0) \oplus S^{+}_{k-1/2}(M,\chi_0)|B_4 \bigoplus S^{+,new}_{k-1/2}(4M,\chi_0)$$

and

$$S_{2k-2}(M,\chi^2) = S_{2k-2}(M/2,\chi^2) \oplus S_{2k-2}(M/2,\chi^2)|B_2.$$

Proposition 3.5 and Proposition 3.8 give

$$S_D : S^{+}_{k-1/2}(4M,\chi_0) \longrightarrow S_{2k-2}(M,\chi^2)$$

and

$$S_D : S^{+}_{k-1/2}(M,\chi_0) \longrightarrow S_{2k-2}(M/2,\chi^2),$$

where we use $S_D = S_{D;k-1/2,M,\chi_0}$ or $S_D = S_{D;k-1/2,4M,\chi_0}$. Let $g \in S^{+}_{k-1/2}(M,\chi_0)$. Since S_D commutes with B_4 and B_2 as

$$g|B_4 S_D = g|S_D B_2,$$

each of the Shimura map S_D maps

$$S^{+}_{k-1/2}(M,\chi_0) \oplus S^{+}_{k-1/2}(M,\chi_0)|B_4$$

into the space

$$S_{2k-2}(M/2,\chi^2) \oplus S_{2k-2}(M/2,\chi^2)|B_2.$$

Since S_D maps $S^{+}_{k-1/2}(4M,\chi_0)$ into $S_{2k-2}(M,\chi^2)$ and the Shimura correspondence preserves eigenclasses with respect to Hecke operators $T_{n^2}, (n,M) = 1$, if $g \in S^{+,new}_{k-1/2}(4M,\chi_0)$ then $g|S_D$ belongs to $S^{new}_{2k-2}(M,\chi^2)$, and hence $g|S_D = 0$ for all such fundamental discriminants D. This proves $g = 0$, if not there exists a fundamental discriminant $D, (D,M) = 1$ with $a_g(|D|) \neq 0$ so that $g|S_D \neq 0$, which is not true. Now the result follows. □

We now get Theorem 1.2 by combining the above results.

4. Theory of newforms of $J_{k,1}^{cusp}(M,\chi)$

4.1. Eichler-Zagier map $\mathcal{Z}_1$

Let $k \geq 2$ be an integer. $\epsilon = (-1)^k$. Let $\mathcal{Z}_1$ denote the Eichler-Zagier map defined in the preliminaries. We prove the following.

$$\mathcal{Z}_1 : J_{k,1}^{cusp}(M,\chi) \longrightarrow S_{k-1/2}^{+}(4M,\chi_0).$$

Let $D < 0$ be a discriminant and $r (\mathrm{mod}\ 2)$ with $r^2 \equiv D (\mathrm{mod}\ 4)$. Let $P_{k,1,M,\chi_0;D,r}$ denote the (D,r)-th Poincaré series in $J_{k,1}^{cusp}(M,\chi)$. Let $P^{+}_{k-1/2,4M,\chi_0;|D|}$ be the $|D|$-th Poincaré series in $S^{+}_{k-1/2}(4M,\chi_0)$ as defined in §3.2.

The Fourier expansion of $P_{k,1,M,\chi_0;D,r}$ is given by (which can be obtained using standard arguments):

$$P_{k,1,M,\chi_0;D,r}(\tau,z) = \sum_{\substack{D',r'\in\mathbb{Z},\\ D'<0}} g^{\pm}_{k,M,\chi_0;D,r}(D',r') e\left(\frac{r'^2 - D}{4}\tau + r'z\right), \tag{8}$$

where $g^{\pm}_{k,1,M,\chi_0;D,r}(D',r')$ is symmetrised or antisymmetrised with respect to r', i.e.,

$$g^{\pm}_{k,1,M,\chi_0;D,r}(D',r') = g_{k,1,M,\chi_0;D,r}(D',r') + \chi(-1)(-1)^k g_{k,1,M,\chi_0;D,r}(D',-r')$$

with $D' = r'^2 - 4n'$, $D = r^2 - 4n$, and

$$g_{k,1,M,\chi_0;D,r}(D',r') = \delta(D,r;D',r') + \pi\sqrt{2} i^{-k} (D'/D)^{k/2-3/4} \times \sum_{c\geq 1} H_{Mc,\chi}(D,r;D',r') J_{k-3/2}\left(\frac{\pi\sqrt{D'D}}{Mc}\right),$$

where

$$\delta(D,r;D',r') = \begin{cases} 1 & \text{if } D' = D, r' \equiv r (\mathrm{mod}\ 2) \\ 0 & \text{otherwise,} \end{cases}$$

$J_{k-3/2}(.)$ is the Bessel function and

$$H_{c,\chi}(D,r;D',r') = \frac{1}{c^{3/2}} \sum_{\substack{\lambda,\delta (\mathrm{mod}\ c),\\ \delta^{-1}\delta\equiv 1 (\mathrm{mod}\ c)}} \overline{\chi}(\delta) e_c(\delta^{-1}(\lambda^2 + r\lambda + n) + n'\delta - r'\lambda) e_{2c}(-rr').$$

The Fourier expansion of $P^{+}_{k-1/2,4M,\chi_0;|D|}$ is given by

$$P^{+}_{k-1/2,4M,\chi_0;|D|}(\tau) = \sum_{\substack{m\geq 1,\\ \epsilon(-1)^{k-1}m\equiv 0,1 \pmod 4}} g^{+}_{k-1/2,4M,\chi_0;|D|}(m)e^{2\pi i m\tau}, \tag{9}$$

where

$$g^{+}_{k-1/2,4M,\chi_0;|D|}(m) = \delta_{|D|,m} + \pi\sqrt{2}(-1)^{\lfloor \frac{k}{2}\rfloor}(1-(-1)^{k-1}i)(m/|D|)^{k/2-3/4} \times \sum_{c\geq 1} H_{4Mc,\chi}(m,|D|)J_{k-3/2}\left(\frac{4\pi\sqrt{m|D|}}{4Mc}\right),$$

$\delta_{n,m}$ is the Kronecker delta, $J_{k-3/2}(.)$ is the Bessel function and

$$H_{4Mc,\chi}(m,n) = \frac{1}{4Mc}\sum_{\substack{\delta \pmod{4Mc},\\ \delta^{-1}\delta\equiv 1 \pmod{4Mc}}} \overline{\chi}_0(\delta)\left(\frac{4Mc}{\delta}\right)\left(\frac{-4}{\delta}\right)^{k-1/2} e_{4Mc}(m\delta+n\delta^{-1}).$$

We have the following standard identity:

Lemma 4.1. *Gauss sum identity: Let* $4|c$, $(c,d)=1$. *Then*

$$\sum_{\lambda \pmod c} e_c(d\lambda^2) = (1+i)\sqrt{c}\left(\frac{c}{d}\right)\left(\frac{-4}{d}\right)^{-1/2}.$$

We prove the Eichler-Zagier map sends Jacobi Poincaré series into the plus space Poincaré series.

Proposition 4.2.

$$P_{k,1,M,\chi_0;D,r}|\mathcal{Z}_1 = 2P^{+}_{k-1/2,4M,\chi_0;|D|}.$$

Proof. To prove the above equation, we compare the $|D'|$-th coefficients on both sides. We need to show that

$$\sum_{\substack{r' \pmod 2,\\ D'\equiv r'^2 \pmod 4}} g^{\pm}_{k,M,\chi_0;D,r}(D',r') = 2g^{+}_{k-1/2,4M,\chi_0;|D|}(|D'|). \tag{10}$$

Comparing the first terms on both sides of the above equation, using (8) and (9) we have

$$\begin{aligned}\delta(D,r;D',r') + \chi(-1)(-1)^k\delta(D,r;D',-r') &= \delta(D,r;D',r') + \delta(D,r;D',r') \\ &= \begin{cases} 2; & D = D' \\ 0; & D \neq D' \end{cases} \\ &= 2\delta_{|D|,|D'|}.\end{aligned}$$

Hence, the first terms on both sides of equation (10) are equal. Now, we compare the second terms on both sides. Second term in the LHS of equation (10) is given by

$$\begin{aligned}&\pi\sqrt{2}i^{-k}(D'/D)^{k/2-3/4} \times \sum_{c\geq 1} H_{Mc,\chi}(D,r;D',r')J_{k-3/2}\left(\frac{\pi\sqrt{D'D}}{Mc}\right) + \\ &\qquad \pi\sqrt{2}i^{-k}(D'/D)^{k/2-3/4} \times \sum_{c\geq 1} H_{Mc,\chi}(D,r;D',-r')J_{k-3/2}\left(\frac{\pi\sqrt{D'D}}{Mc}\right) \\ &= 2.\pi\sqrt{2}i^{-k}(D'/D)^{k/2-3/4} \sum_{c\geq 1} H_{Mc,\chi}(D,r;D',r')J_{k-3/2}\left(\frac{\pi\sqrt{D'D}}{Mc}\right).\end{aligned}$$

Second term in the RHS of equation (10) is given by

$$\begin{aligned}&2.\pi\sqrt{2}(-1)^{\lfloor \frac{k}{2}\rfloor}(1-(-1)^{k-1}i)(|D'|/|D|)^{k/2-3/4} \\ &\quad \times \sum_{c\geq 1} H_{4Mc,\chi}(|D'|,|D|)J_{k-3/2}\left(\frac{4\pi\sqrt{|D'||D|}}{4Mc}\right).\end{aligned}$$

Note that $(-1)^{\lfloor k/2\rfloor}(1-(-1)^{k-1}i) = i^{-k}(1+i)$. Hence,

$$\begin{aligned}&i^{-k}H_{Mc,\chi}(D,r;D',r') \\ &\quad = \frac{i^{-k}}{(Mc)^{3/2}} \sum_{\substack{\lambda,\delta \pmod{Mc}, \\ \delta^{-1}\delta\equiv 1 \pmod{Mc}}} \overline{\chi}(\delta)e_{Mc}(\delta^{-1}(\lambda^2+r\lambda+n)+n'\delta-r'\lambda)e_{2Mc}(-rr').\end{aligned}$$

Using $\lambda \mapsto \delta\lambda$ the above equals

$$\begin{aligned}&\frac{i^{-k}}{(Mc)^{3/2}} \sum_{\substack{\lambda,\delta \pmod{Mc}, \\ \delta^{-1}\delta\equiv 1 \pmod{Mc}}} \overline{\chi}(\delta)e_{Mc}(\delta\lambda^2+r\lambda+n\delta^{-1}+n'\delta-r'\delta\lambda)e_{2Mc}(-rr') \\ &\quad = \frac{i^{-k}}{(Mc)^{3/2}} \sum_{\delta \pmod{Mc}} \overline{\chi}(\delta)\left(\sum_{\lambda \pmod{Mc}} e_{Mc}(\delta\lambda^2+(r-r'\delta)\lambda)\right) \\ &\qquad \times e_{Mc}(n\delta^{-1}+n'\delta)e_{2Mc}(-rr')\end{aligned}$$

$$= \frac{i^{-k}}{(Mc)^{3/2}} \sum_{\substack{\delta (\mathrm{mod}\ Mc), \\ \delta^{-1}\delta \equiv 1 (\mathrm{mod}\ Mc)}} \overline{\chi}(\delta) \left(\sum_{\lambda (\mathrm{mod}\ Mc)} e_{Mc} \left(\delta \left(\lambda + \frac{r\delta^{-1} - r'}{2} \right)^2 \right) \right)$$

$$\times e_{4Mc}(-r^2\delta^{-1} + 2rr' - r'^2\delta) e_{4Mc}(4n\delta^{-1} + 4n'\delta) e_{4Mc}(-2rr')$$

$$= \frac{i^{-k}}{(Mc)^{3/2}} \sum_{\substack{\delta (\mathrm{mod}\ Mc), \\ \delta^{-1}\delta \equiv 1 (\mathrm{mod}\ Mc)}} \overline{\chi}(\delta) \left((1+i) \left(\frac{-4}{\delta} \right)^{-1/2} \left(\frac{Mc}{\delta} \right) \sqrt{Mc} \right)$$

$$\times e_{4Mc}(|D|\delta^{-1} + |D'|\delta) \quad \text{(using Lemma 4.1)}$$

$$= \frac{i^{-k}(1+i)}{Mc} \sum_{\substack{\delta (\mathrm{mod}\ Mc), \\ \delta^{-1}\delta \equiv 1 (\mathrm{mod}\ Mc)}} \overline{\chi}(\delta) \left(\frac{4(-1)^k}{\delta} \right) \left(\frac{-4}{\delta} \right)^{k-1/2} \left(\frac{Mc}{\delta} \right) e_{4Mc}(|D|\delta^{-1} + |D'|\delta)$$

$$= \frac{i^{-k}(1+i)}{4Mc} \sum_{\substack{\delta (\mathrm{mod}\ 4Mc), \\ \delta^{-1}\delta \equiv 1 (\mathrm{mod}\ 4Mc)}} \overline{\chi}_0(\delta) \left(\frac{-4}{\delta} \right)^{k-1/2} \left(\frac{4Mc}{\delta} \right) e_{4Mc}(|D|\delta^{-1} + |D'|\delta)$$

$$= (-1)^{\lfloor k/2 \rfloor}(1 - (-1)^{k-1} i) H_{4Mc,\chi}(|D'|, |D|).$$

So, the second terms on both sides of equation (10) are also equal. This completes the proof. □

We now state a result which relates the Petersson scalar products in $J_{k,1}^{cusp}(M,\chi)$ and $S_{k-1/2}^{+}(4M,\chi)$ whose proof follows by using the same argument of the Proposition 5.1 in [8]. So we omit the details.

Lemma 4.3. *We have*

$$\langle \phi | \mathcal{Z}_1, \psi | \mathcal{Z}_1 \rangle = c \langle \phi, \psi \rangle$$

for all $\phi, \psi \in J_{k,1}^{cusp}(M,\chi)$ *with*

$$c = 2 \frac{i_M}{4^{k-3/2}}.$$

Hence, $\mathcal{Z}_1$ *is a canonical isomorphism which preserves the Hecke eigenforms and the Petersson scalar product structures.*

4.2. Decomposition of the space of Jacobi forms

We start with the following lemma.

Lemma 4.4. *Suppose* $4|D$. *Let* $D/4 \equiv 0, 1 (\text{mod } 4)$. *Then,*

$$P^{+}_{k-1/2,M,\chi_0;\frac{|D|}{4}} = P^{+}_{k-1/2,4M,\chi_0;|D|}|U_4$$

Proof. It is enough to show that m-th Fourier coefficients are equal for all integers $m \geq 1$. They are respectively given by

$$g^{+}_{k-1/2,M,\chi_0;\frac{|D|}{4}}(m) = \delta_{\frac{|D|}{4},m} + \pi\sqrt{2}(-1)^{\lfloor \frac{k}{2} \rfloor}(1-(-1)^{k-1}i)(4m/|D|)^{k/2-3/4} \times \sum_{c\geq 1} H_{Mc,\chi}(m, |D|/4) J_{k-3/2}\left(\frac{4\pi\sqrt{m\frac{|D|}{4}}}{Mc}\right)$$

and

$$g^{+}_{k-1/2,4M,\chi_0;|D|}(4m) = \delta_{|D|,4m} + \pi\sqrt{2}(-1)^{\lfloor \frac{k}{2} \rfloor}(1-(-1)^{k-1}i)(4m/|D|)^{k/2-3/4} \times \sum_{c\geq 1} H_{4Mc,\chi}(4m, |D|) J_{k-3/2}\left(\frac{4\pi\sqrt{4m|D|}}{4Mc}\right).$$

Since $4|D$, we have

$$\delta_{\frac{|D|}{4},m} = \delta_{|D|,4m}$$

and hence the first terms are equal. The second terms are also equal since

$$H_{Mc,\chi}(m, |D|/4) = H_{4Mc,\chi}(4m, |D|),$$

which follows by using their definitions. $\square$

Lemma 4.5. *Let* $4|D$, $D/4 \equiv 0, 1 (mod\, 4)$. *Then,*

$$P_{k,1,M,\chi_0;D,r}|U_J(4)Z_1 = 2P^{+}_{k-1/2,M,\chi_0;\frac{|D|}{4}}$$

Proof. Follows from Proposition 4.2 and Lemma 4.4. $\square$

Lemma 4.6. $S^{+}_{k-1/2}(M, \chi_0)$ *is spanned by all the Poincaré series* $P^{+}_{k-1/2,M,\chi_0;\frac{|D|}{4}}$, *where* D *varies over all the discriminants with* $4\big||D|$ *and* $D/4 \equiv 0, 1 (\text{mod } 4)$.

Proof. Let $f \in S^{+}_{k-1/2}(M, \chi_0)$ such that it is in the orthogonal complement of the sub space of $S^{+}_{k-1/2}(M, \chi_0)$ spanned by the Poincaré series $P^{+}_{k-1/2,M,\chi_0;\frac{|D|}{4}}$, where D varies over all the discriminants with $4\big||D|$ and $D/4 \equiv 0, 1 (\text{mod } 4)$. Then,

$$\langle f, P^{+}_{k-1/2,M,\chi_0;\frac{|D|}{4}} \rangle = 0,$$

for all $4\big||D|$ with $D/4 \equiv 0, 1 (\bmod\ 4)$. This implies that

$$a_f \left(\frac{|D|}{4} \right) = 0,$$

for all $D \equiv 0, 1 (\bmod\ 4)$, or equivalently

$$a_{f|B_4}(|D|) = 0,$$

for all $D \equiv 0, 1 (\bmod\ 4)$. Therefore, $f|B_4 = 0$ (or) $f = 0$. Hence, the result follows. □

We now define the space of newforms in $J_{k,1}^{cusp}(M, \chi)$ by

$$J_{k,1}^{cusp;new}(M, \chi) = \text{the linear span of the set} \{P_{k,1,M,\chi;D,r}|U_J(4)\},$$

where D varies over all the discriminants with $4\big||D|$ and $D/4 \equiv 0, 1 (\bmod\ 4)$. A Hecke eigenform which is a simultaneous eigenform for Hecke operators $T_J(n), (n, M) = 1$ in $J_{k,1}^{cusp;new}(M, \chi)$ is called a newform. We have the image of Jacobi newforms under the Eichler-Zagier isomorphism:

Lemma 4.7. $J_{k,1}^{cusp;new}(M, \chi)|\mathcal{Z}_1 = S_{k-1/2}^{+}(M, \chi_0)$.

Proof. Follows from Lemma 4.5 and Lemma 4.6. □

We now give the proof of Theorem 1.3.

Proof of Theorem 1.3. From Theorem 1.2, we have the decomposition

$$S_{k-1/2}^{+}(4M, \chi_0) = S_{k-1/2}^{+}(M, \chi_0) \bigoplus S_{k-1/2}^{+}(M, \chi_0)|B_4.$$

Also, we have

$$\phi|B_J(4)\mathcal{Z}_1 = \phi|\mathcal{Z}_1 B_4,$$

where $\phi \in J_{k,1}^{cusp}(M, \chi)$. Using these we get

$$\begin{aligned} J_{k,1}^{cusp}(M, \chi)|\mathcal{Z}_1 &= S_{k-1/2}^{+}(4M, \chi_0) \\ &= S_{k-1/2}^{+}(M, \chi_0) \bigoplus S_{k-1/2}^{+}(M, \chi_0)|B_4 \\ &= J_{k,1}^{cusp;new}(M, \chi)|\mathcal{Z}_1 \bigoplus J_{k,1}^{cusp;new}(M, \chi)|B_J(4)|\mathcal{Z}_1 \\ &= \left(J_{k,1}^{cusp;new}(M, \chi) \bigoplus J_{k,1}^{cusp;new}(M, \chi)|B_J(4) \right) |\mathcal{Z}_1. \end{aligned}$$

Thus, by using $\mathcal{Z}_1$ defines an isomorphism from $J_{k,1}^{cusp}(M, \chi)$ into $S_{k-1/2}^{+}(4M, \chi_0)$, we have

$$J_{k,1}^{cusp}(M,\chi) = J_{k,1}^{cusp;new}(M,\chi) \bigoplus J_{k,1}^{cusp;new}(M,\chi)|B_J(4).$$

The multiplicity one result holds on $J_{k,1}^{cusp;new}(M,\chi)$, which follows from Theorem 1.1 and the isomorphism $\mathcal{Z}_1$. $\square$

5. Theory of newforms of $\mathcal{S}_k^*(\Gamma_0^2(M),\chi)$ and Saito-Kurokawa isomorphism

Let $\mathcal{S}_k(\Gamma_0^2(M),\chi)$ be the space of Siegel cusp forms of weight k, level M, genus 2 and character χ. Let $\mathcal{S}_k^*(\Gamma_0^2(M),\chi)$ denote the Maass space in $\mathcal{S}_k(\Gamma_0^2(M),\chi)$. Let $V_{m,\chi}$ be the index shifting operator

$$V_{m,\chi} : J_{k,1}^{cusp}(M,\chi) \to J_{k,m}^{\text{cusp}}(M,\chi)$$

(as per [6]). If

$$\phi(\tau,z) = \sum_{\substack{D<0, r\in\mathbb{Z},\\ D\equiv r^2 \pmod 4}} c(D,r) e\left(\frac{r^2-D}{4}\tau + rz\right) \in J_{k,1}^{cusp}(M,\chi),$$

then

$$\phi|V_{m,\chi}(\tau,z) = \sum_{\substack{D<0, r\in\mathbb{Z},\\ D\equiv r^2 \pmod{4m}}} \left(\sum_{\substack{d|(r,m),\,(d,M)=1,\\ D\equiv r^2 \pmod{4md}}} \overline{\chi(d)} d^{k-1} c\left(\frac{D}{d^2},\frac{r}{d}\right) e\left(\frac{r^2-D}{4m}\tau + rz\right)\right).$$

For $\phi \in J_{k,1}^{cusp}(M,\chi)$, we define the Maass embedding $\iota_{M,\chi}$ as follows (see [6]):

$$\phi|\iota_{M,\chi} = \sum_{m\geq 1} (\phi|_{k,1} V_{m,\chi})(\tau,z) e^{2\pi i m w}.$$

Then we have the following result:

Proposition 5.1 *([6], Theorem 3.2). The map $\iota_{M,\chi}$ gives an embedding of $J_{k,1}^{cusp}(M,\chi)$ into $\mathcal{S}_k^*(\Gamma_0^2(M),\chi)$.*

In ([5], corollary 4.2) it was proved that the Fourier coefficients of the forms in $\mathcal{S}_k^*(\Gamma_0^2(M),\chi)$ also satisfy certain relations analogous to the classical Maass relation. The converse part was also proved; i.e., if the coefficients of $F \in \mathcal{S}_k(\Gamma_0^2(M),\chi)$ satisfy the said relations, then $F \in \mathcal{S}_k^*(\Gamma_0^2(M),\chi)$.

For $F \in \mathcal{S}_k^*(\Gamma_0^2(M),\chi)$, define the operator $B_S(4)$ by

$$F(\tau,z,\tau')|B_S(4) = F(4\tau,4z,4\tau').$$

For $\phi \in J_{k,1}^{cusp;new}(M,\chi)$, we have $\phi|\iota_{M,\chi}|B_S(4) = \phi|B_J(4)|\iota_{M,\chi}$. Let

$$\mathcal{S}_k^{*;new}(\Gamma_0^2(M),\chi) = J_{k,1}^{cusp;new}(M,\chi)|\iota_{M,\chi}.$$

5.1. Saito-Kurokawa lift

Let N be an odd integer and $M = 2^{\alpha-2}N$, where $\alpha > 6$. Let χ be a primitive Dirichlet character modulo M. Let $T_S(p), T'_S(p)$ (for prime $p \nmid M$) and $U_S(p)$ (for prime $p|M$) denote standard Hecke operators in $\mathcal{S}_k(\Gamma_0^2(M),\chi)$ (see §4 of [6]). An eigenform in $\mathcal{S}_k(\Gamma_0^2(M),\chi)$ under the above operators is called a Hecke eigenform. A non-zero Hecke eigenform which belongs to $\mathcal{S}_k^{*;new}(\Gamma_0^2(M),\chi)$ is called a newform in the Maass space. For a newform $F \in \mathcal{S}_k^{*;new}(\Gamma_0^2(M),\chi)$, let γ_p, ω_p and μ_p denote the corresponding eigenvalues with respect to $T_S(p), T'_S(p), U_S(p)$ respectively. Similarly, let $T_J(p)$ (for prime $p \nmid M$) and $U_J(p)$ (for prime $p|M$) denote standard Hecke operators in $J_{k,1}^{cusp}(M,\chi)$, and let $a_f(p)$ denote the corresponding eigenvalues for the newform $\phi \in J_{k,1}^{cusp;new}(M,\chi)$ which corresponds to the normalised Hecke eigenform $f \in S_{2k-2}(M/2,\chi^2)$. Let $F = \phi|\iota_{M,\chi}$ (as in §5). Then, theorem 4.1 of [6] gives

$$\left.\begin{aligned} \gamma_p &= a_f(p) + \chi(p)(p^{k-2} + p^{k-1}), \quad p \nmid M \\ \omega_p &= \chi(p)(p^{k-2} + p^{k-1})a_f(p) + \chi(p^2)(2p^{2k-3} + p^{2k-4}), \quad p \nmid M \\ \mu_p &= a_f(p), \quad p \mid M. \end{aligned}\right\} \tag{11}$$

For a Hecke eigenform $F \in \mathcal{S}_k^{*;new}(\Gamma_0^2(M),\chi)$, the Andrianov zeta function $Z_F(s)$ has the Euler product expansion (see [6])

$$Z_F(s) = \prod_{p|M}(1-\mu_p p^{-s})^{-1}\prod_{p\nmid M} Q_p(p^{-s})^{-1},$$

where

$$\begin{aligned} Q_p(p^{-s}) = 1 - \gamma_p p^{-s} + (p\omega_p + (p^2+1)\chi(p^2)p^{2k-5})p^{-2s} - \omega_p\chi(p^2)p^{2k-3-3s} \\ + \chi(p^4)p^{4k-6-4s}. \end{aligned}$$

This, combined with (11) give

$$\begin{aligned} Z_F(s) &= L(s-k+1,\chi)L(s-k+2,\chi)\prod_{p|M}(1-a_f(p)p^{-s})^{-1}\prod_{p\nmid M}(1-a_f(p)p^{-s} \\ &\quad + \chi(p^2)p^{2k-3-2s})^{-1} \\ &= L(s-k+1,\chi)L(s-k+2,\chi)L(f,s), \end{aligned}$$

where $L(f,s)$ is the L-function of f given by

$$L(f,s) = \sum_{n\geq 1} a_f(n)n^{-s}.$$

Summarizing the above, we obtain the stated Theorem 1.4.

Finally, we make the following observation:

Remark 5.2. The map $\mathcal{Z}_1$ is a canonical isomorphism from $J_{k,1}^{cusp}(M,\chi)$ into the space $S_{k-1/2}^{+}(4M,\chi_0)$, which has the decomposition:

$$S_{k-1/2}^{+}(4M,\chi_0) = S_{k-1/2}^{+}(M,\chi_0) \oplus S_{k-1/2}^{+}(M,\chi_0)|B_4.$$

Hence, we have the corresponding decomposition for Jacobi forms as

$$J_{k,1}^{cusp}(M,\chi) = J_{k,1}^{cusp,new}(M,\chi) \oplus J_{k,1}^{cusp,new}(M,\chi)|B_J(4).$$

This gives, if $\phi \in J_{k,1}^{cusp,new}(M,\chi)$, then we have $\phi|\mathcal{Z}_1 \in S_{k-1/2}^{+}(M,\chi_0)$ and $\phi|B_J(4)\mathcal{Z}_1 = \phi|\mathcal{Z}_1 B_4 \in J_{k,1}^{cusp}(M,\chi)$. Now by using the space $S_{k-1/2}^{+}(M,\chi_0)$ is isomorphic to the space $S_{2k-2}(M/2,\chi^2)$ as module over Hecke algebra and using the embedding of $J_{k,1}^{cusp}(M,\chi)$ in the Maass space, we realise that a normalised Hecke eigenform $f \in S_{2k-2}(M/2,\chi^2)$ is lifted into two linearly independent Hecke eigenforms F and $F|B_S(4)$, where $F \in \mathcal{S}_k^{*,new}(\Gamma_0^2(M),\chi)$. We call the form $F \in \mathcal{S}_k^{*,new}(\Gamma_0^2(M),\chi)$ a newform of weight k, level M, character χ in the Maass space.

Data availability

No data was used for the research described in the article.

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